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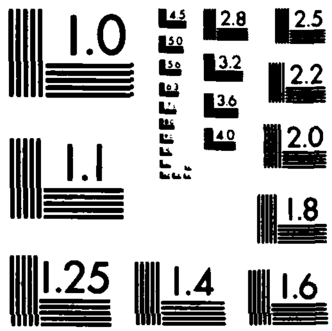
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CONVEX-ORDERING AMONG FUNCTIONS, WITH APPLICATIONS TO RELIABILITY AND MATHEMATICAL STATISTICS

by

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ABSTRACT

Hardy, Littlewood and Pólya (1934) introduced the notion of one function being convex with respect to a second function and developed some inequalities concerning the means of the functions. We use this notion to establish a partial order called convex-ordering among functions. In particular, the distribution functions encountered in many parametric families in reliability theory are convex-ordered. We have formulated some inequalities which can be used for testing whether a sample comes from F or G, when F and G are within the same convex family. Performance characteristics of different coherent structures can also be compared with respect to this partial ordering. For example, we will show that the reliability of a k+1-out-of-n system is convex with respect to the reliability of a k-out-of-n system.

When F is convex with respect to G, the tail of the distribution F is heavier than that of G; therefore, our convex ordering implies stochastic ordering. The ordering is also related to total positivity and monotone likelihood ratio families. This provides us a tool to obtain some useful results in reliability and mathematical statistics.

1. Introduction.

For any distribution function F, we define the inverse of F by $F^{-1}(t) = \inf\{x: F(x) \geq t\}$.

Several notions of partial ordering among life distributions have been established earlier by various authors. Van Zwet (1964) introduced the notion: F is convex-ordered with respect to G if $G^{-1}F$ is convex on the support of F. The basis of this ordering between distribution functions, and hence between random variables, is that one random variable can be expressed as a convex transformation of another random variable. Barlow and Proschan (1966) have derived tolerance limits for the distributions which are ordered in the sense of Van Zwet.

In 1981, Larry Lee defined and analyzed the following notion of convex ordering: F is convex-hazard ordered with respect to G, written $F \underset{CH}{<} G$ if $R_F R_G^{-1}$ is convex, where $R_F = -\log \bar{F}$ is the hazard function of survival function $\bar{F} = 1-F$. Lee used this convexity property to generalize certain inequalities and preservation theorems in reliability. We define the following notion of convex ordering, which is different from those of Van Zwet (1964) or Lee (1981).

1.1 Definition. For any distributions F and G, we say that F is more convex than G written $F \overset{c}{>} G$, if FG^{-1} is a convex function in the interval $[0,1]$.

The notion we propose here permits one distribution function to be expressed as a convex transformation of another distribution function. E.g., x^3 is more convex than x^2 on the interval $[0,1]$. This concept coincides with that of Hardy, Littlewood and Pólya (1934, p.65). Although the above definition of this ordering applies to the class of all monotonic functions, we shall restrict our attention mainly to life distributions. Note that in the class of increasing functions, a necessary and sufficient condition that f be convex is that $f \overset{c}{>} g$ for some convex function g.

A very useful way to characterize this convex ordering using the densities of the distributions is given in the following theorem.

<u>1.2. Theorem</u>. Let the distribution functions F and G be absolutely continuous with densities f and g. In order that $F \overset{c}{>} G$, it is necessary and sufficient that $\frac{f}{g}$ be increasing.

<u>Proof</u>. FG^{-1} is convex if and only if its derivative $\frac{fG^{-1}(t)}{gG^{-1}(t)}$ is increasing in t. By noting that G^{-1} is an increasing function, we obtain the conclusion.||

Using this result, we easily see that this convex ordering represents a partial ordering on the class of distributions with continuous densities. Specifically, convex ordering has the following properties:

(a) Reflexive: $F \overset{c}{>} F$.

(b) Transitive: $F \overset{c}{>} G$ and $G \overset{c}{>} H$ imply $F \overset{c}{>} H$.

(c) Antisymmetric: $F \overset{c}{>} G$ and $G \overset{c}{>} F$ imply $F = G$.

We now define the notion of a convex-ordered family.

<u>1.3. Definition.</u> A family of distributions $\{F_\alpha\}$ is said to be a convex-ordered family, or simply a convex family if $F_{\alpha_2} \overset{c}{>} F_{\alpha_1}$ for $\alpha_2 > \alpha_1$.

The following families of distributions are ccnvex ordered with respect to α for $\alpha > 0$.

Examples

(1) Exponential - $F_\alpha(t) = 1-e^{-t/x}$, $t > 0$.

(2) Gamma - $F_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-x}\, dx$, $t > 0$.

(3) Truncated Normal - $F_\alpha(t) = \frac{1}{a_\alpha \sigma\sqrt{2\pi}} \int_0^t e^{-(x-\alpha)^2/2\sigma^2}\, dx$ for $t > 0$, where $\sigma > 0$ is fixed and $a_\alpha = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\alpha)^2/2\sigma^2}\, dx$.

(4) Weibull - $F_\alpha(t) = 1-e^{-(t/\alpha)^\lambda}$ for $t > 0$, where $\lambda > 0$ is fixed.

(5) Proportional hazards - $F_\alpha(t) = 1-e^{-\frac{1}{\alpha}R(t)}$, $t > 0$, where $R(t) = -\log \bar{F}(t)$ is the hazard function of some life distribution F.

The theory of total positivity has been used to obtain many new results in reliability and life testing. In studying convex families, we can also make use of this powerful tool. The main results of total positivity can be found in Karlin (1968).

1.4. Definition. A nonnegative function $f_\alpha(x)$ on $R \times R$ is totally positive of order 2 (TP_2) in (α,x) if

$$\begin{vmatrix} f_{\alpha_1}(x_1) & f_{\alpha_1}(x_2) \\ f_{\alpha_2}(x_1) & f_{\alpha_2}(x_2) \end{vmatrix} \geq 0$$

for all $\alpha_1 < \alpha_2$ and $x_1 < x_2$ (also called the monotone likelihood ratio property.)

The next theorem relates total positivity to convex ordering.

1.5. Theorem. $\{F_\alpha\}$ is a convex family if and only if the corresponding density $f_\alpha(t)$ is TP_2 in (α,t).

Proof. By Theorem 1.2, we have that $F_{\alpha_2} \overset{c}{>} F_{\alpha_1}$ for $\alpha_2 > \alpha_1$ if and only if $\frac{f_{\alpha_2}}{f_{\alpha_1}}$ is increasing. Thus for $\alpha_2 > \alpha_1$, $t_2 > t_1$, $\begin{vmatrix} f_{\alpha_1}(t_1) & f_{\alpha_1}(t_2) \\ f_{\alpha_2}(t_1) & f_{\alpha_2}(t_2) \end{vmatrix} \geq 0$, which is the defining condition that $f_\alpha(t)$ is TP_2 in (α,t).||

An immediate consequence of this theorem is the following corollary.

1.6. Corollary. If $\{F_\alpha\}$ is a convex family, then $F_\alpha(t)$ is TP_2 in (α,t).

Proof. By Theorem 1.5, $f_\alpha(x)$ is TP_2 in (α,x).

It follows from the basic composition formula of Pólya and Szegö (1925) that

$$F_\alpha(t) = \int_{-\infty}^{t} f_\alpha(x)\,dx \text{ is } TP_2 \text{ in } (\alpha,t).\|$$

Another characterization of convex ordering is given by

1.7. Theorem. $F \overset{c}{>} G$ if and only if $\bar{G} \overset{c}{>} \bar{F}$.

Proof. Taking the derivative of $\bar{G}\bar{F}^{-1}$, we can see that

$\bar{G} \overset{c}{>} \bar{F}$ iff $\frac{g\bar{F}^{-1}(t)}{f\bar{F}^{-1}(t)}$ is increasing in t

iff $\frac{g}{f}$ is decreasing [since $\bar{F}^{-1}(t)$ is decreasing in t]

iff $F \overset{c}{>} G$ by Theorem 1.2.||

We end this section with the following comparisons of our convex ordering and those of Van Zwet and Lee. We note that the Weibull family is not convex ordered according to the shape parameter λ, but it is convex ordered in the sense of Van Zwet. Let $F_1(t) = t^2$ on the interval $[0,1]$ and $F_2(t) = 1-\sqrt{1-t^2}$ on the same interval. Then F_2 is more convex than F_1. Since $F_1^{-1}F_2(t) = (1 - \sqrt{1-t^2})^{\frac{1}{2}}$ is not convex, F_2 is not convex-ordered with respect to F_1.

One way to describe F more convex than G, is to say that the tail of distribution F is heavier than that of distribution G. Suppose F and G are the distribution functions of the random variables X and Y. Then by Corollary 1.6, $\frac{F}{G}$ is an increasing function. Hence $F(x) \le G(x)$ holds for each x, the defining condition for x to be stochastically larger than Y. Thus convex ordering implies stochastic ordering. Next, we will show that $F \overset{c}{>} G$ also implies that the hazard rate $r_F = \frac{f}{\bar{F}}$ of F is less than that of G. Comparing this result with the convex-hazard order of Lee (1981), which requires that $\frac{r_F}{r_G}$ be an increasing function of t, we see that neither notion of convex ordering implies the other.

__1.8. Corollary__. If $F \overset{c}{>} G$, then $r_F(t) \le r_G(t)$ for all t.

__Proof__. By Theorem 1.5, $f(t_1)g(t) \le f(t)g(t_1)$, for all $t_1 \le t$. Integrating this over $[t_1,\infty)$, we have

$$f(t_1)\bar{G}(t_1) \le \bar{F}(t_1)g(t_1) \quad \text{for all } t_1. \|$$

2. Preservation of Convex Ordering under Operations.

In this section, we show that our notion of convexity is preserved under various standard statistical operations.

First, we show that convex ordering is preserved under mixture of distributions.

__2.1. Theorem__. If $F_\alpha \overset{c}{>} G$ for each α, then $\int F_\alpha d\mu(\alpha) \overset{c}{>} G$ for any mixing distribution μ.

Proof. By Theorem 1.2, $\frac{f_\alpha}{g}$ is increasing for each α. Thus $\frac{\int f_\alpha d\mu(\alpha)}{g}$ is an increasing function.

Again by Theorem 1.2, we have $\int F_\alpha d\mu(\alpha) \overset{c}{>} G.$ ||

A similar proof holds for

2.2. Theorem. If $F \overset{c}{>} G_\alpha$ for each α, then $F \overset{c}{>} \int G_\alpha d\nu(\alpha)$ for any mixing distribution ν.

From Theorems 2.1 and 2.2, we have

2.3. Theorem. If $F_\alpha \overset{c}{>} G_\beta$ for each pair (α,β), then $\int F_\alpha d\mu(\alpha) \overset{c}{>} \int G_\beta d\nu(\beta)$ for any mixing distributions μ and ν.

It should be noted that the condition in Theorem 2.3 cannot be weakened to $F_\alpha \overset{c}{>} G_\alpha$ for each α, as shown in the following example.

2.4. Example.

Let $\bar{F}_1(t) = e^{-t/1.1}$,

$\bar{F}_2(t) = e^{-t/5.1}$,

$\bar{G}_1(t) = e^{-t}$,

$\bar{G}_2(t) = e^{-t/5}$.

Then $F_1 \overset{c}{>} G_1$ and $F_2 \overset{c}{>} G_2$. Define $h(t) = \frac{f_1(t) + f_2(t)}{g_1(t) + g_2(t)}$, where f_i and g_i denote the respective densities of F_i and G_i.

In particular, $h(4.6) = 1.04 > 1.026 = h(6)$. By Theorem 1.2, $\frac{1}{2}(F_1 + F_2) \overset{c}{>} \frac{1}{2}(G_1 + G_2)$ is shown to be false.

Next, we show that convex ordering is preserved under formation of parallel systems of independent components. We begin with some basic definitions and notation from reliability.

Consider n independent components, each of which is either functioning or not. We use the binary variable x_i to indicate the state of the i^{th} component:

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{otherwise.} \end{cases}$$

The state of a system composed of these components is determined by the states of the components. The function $\phi(x_1, \ldots, x_n)$ is called the <u>structure function</u> of the system and is defined by

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if system is functioning,} \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example</u>. A k-out-of-n system functions if and only if at least k out of the n components function. The structure function is given by

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

The i^{th} component is <u>irrelevant</u> to the structure ϕ if ϕ is constant in x_i. We consider <u>monotone</u> systems, that is, systems for which $\phi(x_1, \ldots, x_n) \geq \phi(y_1, \ldots, y_n)$ whenever $x_i \geq y_i$ for all $i = 1, \ldots, n$. If a monotone system has no irrelevant components, it is said to be a <u>coherent</u> system.

Let $P(X_i = 1) = p_i$ denote the reliability of the i^{th} component; the system reliability is given by

$$h_\phi(p_1, \ldots, p_n) = P(\phi(X_1, \ldots, X_n) = 1).$$

Denote the life distribution of the i^{th} component by F_i; then the life distribution F_ϕ of the system is given by

$$F_\phi(t) = 1 - h_\phi(\bar{F}_1(t), \ldots, \bar{F}_n(t)).$$

As a special case, we will consider a parallel structure of n components, i.e., a 1-out-of-n system. The life distribution of the system is given by the product $\Pi_{i=1}^{n} F_i(t)$. The following theorem shows that convex ordering is preserved under formation of parallel systems.

<u>2.5. Theorem</u>. Suppose $F_i \overset{c}{>} G_j$ for each pair (i,j). Then $\Pi_{i=1}^{n} F_i \overset{c}{>} \Pi_{i=1}^{n} G_i$.

<u>Proof</u>. It suffices to prove the theorem for $n = 2$. By Theorems 1.2 and 1.6, for each i and j, $\frac{f_i}{g_j}$, $\frac{F_i}{G_j}$ are increasing functions. Thus $\frac{f_1F_2 + f_2F_1}{g_1G_2 + g_2G_1}$ is increasing. Again, by Theorem 1.2, we have proved the result $F_1F_2 \overset{c}{>} G_1G_2$. ||

When all components are identical, the following theorem shows among other results, the reliability of a k-out-of-n system is more convex than the reliability of a (k-1)-out-of-n system.

<u>2.6. Theorem</u>. Let $h_{n,k}(p)$ be the reliability function of a k-out-of-n system with identical components. Then $h_{n,k+1} \overset{c}{>} h_{n+1,k+1} \overset{c}{>} h_{n,k} \overset{c}{>} h_{n+1,k}$.

<u>Proof</u>.

$$h_{n,k}(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \int_0^p t^{k-1} (1-t)^{n-k} dt$$

when written as the incomplete Beta function. Taking the derivative of $h_{n,k}$, we have

$$h'_{n,k}(p) = \frac{n!}{(n-k)!\,(k-1)!} p^{k-1}(1-p)^{n-k} ,$$

and so,

$$\frac{h'_{n,k}(p)}{h'_{n+1,k}(p)} = \frac{n-k+1}{n+1}\,\frac{1}{1-p}$$

is increasing in p, establishing that $h_{n,k} \overset{c}{>} h_{n+1,k}$. The remaining inequalities can be proved similarly.||

Since the distribution of the k^{th} order statistic corresponds to the life distribution of a k-out-of-n system of identical components, the following corollary is essentially a restatement of Theorem 2.6.

<u>2.7. Corollary</u>. Let $F_{n,k}$ be the distribution of the k^{th} order statistic in a sample of size n from F. Then $F_{n,k+1} \overset{c}{>} F_{n+1,k+1} \overset{c}{>} F_{n,k} \overset{c}{>} F_{n+1,k}$.

A series structure functions if and only if each component functions; i.e., it is a n-out-of-n system. We now show that convex ordering is preserved under formation of series systems with independent components. More generally, we prove the following theorem.

<u>2.8. Theorem</u>. For a k-out-of-n system of independent components, let F_i denote the distribution of the i^{th} component. If $F_i \overset{c}{>} G_j$ for each pair of (i,j), then $F_{n,k} \overset{c}{>} G_{n,k}$.

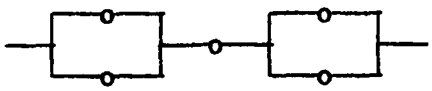
<u>Proof</u>. For a k-out-of-n system, the density is

$$f_{n,k} = \frac{1}{(k-1)!\,(n-k)!} \sum f_{\alpha_1}\bar{F}_{\alpha_2} \cdots \bar{F}_{\alpha_k} F_{\alpha_{k+1}} \cdots F_{\alpha_n} ,$$

where the summation is taken over all permutations of the integers 1,2, ..., n. From Corollary 1.8, $\bar{F}_i g_j \geq f_i \bar{G}_j$ for each pair of (i,j), hence $\frac{\bar{F}_i}{\bar{G}_j}$ is an increasing function. Theorems 1.2 and 1.6 then imply that $\frac{f_{n,k}}{g_{n,k}}$ is increasing. Again by Theorem 1.2, we conclude that $F_{n,k} \overset{c}{>} G_{n,k}$.||

In view of Theorems 2.5 and 2.8, one might ask whether convex ordering is preserved under formation of coherent systems. The example below shows that this is not true in general.

Example. Consider the coherent structure of identical components presented in the following diagram.



The survival distribution of this system is

$$\bar{F}_\phi = \bar{F}^3(2 - \bar{F})^2.$$

For $\bar{F}(t) = e^{-t}$ and $\bar{G}(t) = e^{-2t}$, we have $F \overset{c}{>} G$; but

$$\frac{f_\phi(t)}{g_\phi(t)} = \frac{e^{3t}(2-e^{-t})\ (6-5e^{-t})}{(2-e^{-2t})\ (6-5e^{-2t})}$$

is not increasing in t.

In order to show that convex ordering is preserved under convolution, we need to consider the class of Pólya frequency densities of order 2 (PF_2).

2.9. Definition. f is a Pólya frequency function of order 2 (PF_2) if for all $\Delta > 0$, $f(x+\Delta)/f(x)$ is decreasing in x, $-\infty < x < \infty$.

An equivalent definition is that $\log f(x)$ is concave. Note that each PF_2 function $f(x)$ defines a TP_2 function, $h(x,y) = f(x-y)$.

The following theorem, due to Ghurye and Wallace (1959), gives a sufficient condition on convex families for preservation of convexity under convolution.

<u>2.10. Theorem</u>. Let $\{F_\alpha\}$ and $\{G_\alpha\}$ be convex families with PF_2 densities. Then $\{F_\alpha * G_\alpha\}$ is a convex family.

Denote the n-fold convolution of F by $F^{(n)}$. Then for life distribution F with log concave density, $F^{(n+1)}$ is more convex than $F^{(n)}$. This is a special case of Theorem 1 in Karlin-Proschan (1960).

<u>2.11. Theorem</u>. Let F be a life distribution with PF_2 density, then $\{F^{(n)}\}$ is a convex family.

3. Application of convex ordering.

Very often in life testing, we do not know the exact form of the distribution, but based on physical evidence, we know something about the properties of the distribution. For example, in situations where a normal distribution is assumed, we might suspect that the tail of the underlying distribution is, in fact, heavier than that of the normal distribution. Therefore we want to test the normal assumption against convex ordered alternatives. In this section, we will present an inequality for convex families and apply this inequality to develop tests of such a hypothesis.

<u>3.1. Theorem.</u> (Hardy, Littlewood and Pólya, p.75.) $F \overset{c}{>} G$ if and only if $F^{-1}(\sum_{i=1}^{n}\lambda_i F(x_i)) \geq G^{-1}(\sum_{i=1}^{n}\lambda_i G(x_i))$ for all x_i and $\lambda_i \geq 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n}\lambda_i = 1$.

Note that $\sum_{i=1}^{n}\lambda_i F(x_i)$ is a weighted average. We now apply this result to hypothesis testing.

<u>3.2. Application</u>. Let $X_1, \ldots, X_n$ be a random sample. Suppose we wish to test:

$$H_0:\ X_1, \ldots, X_n \sim G \text{ (known)}$$

against the alternative

$$H_1: \ X_1, \ldots, X_n \sim F, \ F \overset{c}{>} G \ \text{ but otherwise unknown.}$$

Then under H_1, $G^{-1}(\frac{1}{n}\sum_{i=1}^n G(X_i)) \le F^{-1}(\frac{1}{n}\sum_{i=1}^n F(X_i))$.

Since $F(X)$ is uniformly distributed on $[0,1]$, $\frac{1}{n}\sum_{i=1}^n F(X_i)$ is the sample mean of a uniform random variable on $[0,1]$, and thus can be estimated by $\frac{1}{2}$. In this case F is an unknown distribution, but the empirical distribution F_n can be computed from $(X_1, \ldots, X_n)$. Our test procedure is to reject H_0 if $G^{-1}(\frac{1}{n}\sum_{i=1}^n G(X_i))$ is sufficiently smaller than $F_n^{-1}(\frac{1}{2})$. Recall that G is known.

4. Convex ordering for symmetric distribution functions.

In this section, we consider convex orderings between symmetric distribution functions: $\bar{F}(x) = F(-x)$ for all x.

4.1. Definition. $F \overset{sc}{>} G$ if F and G are symmetric distribution functions and $F \overset{c}{>} G$ on $[0,\infty)$, i.e., FG^{-1} is concave-convex about the origin.

Examples of such ordered distributions are:

1) Normal.

Let F_α be the distribution function of $N(0,\alpha^2)$, $\alpha > 0$.

Then $\alpha_2 > \alpha_1 \Rightarrow F_{\alpha_2} \overset{sc}{>} F_{\alpha_1}$.

2) Double exponential.

Let f_α be the density function given by

$$f_\alpha(x) = \frac{1}{2\alpha} e^{-|x|/\alpha}, \ \alpha > 0, \ -\infty < x < \infty .$$

Then $\alpha_2 > \alpha_1 \Rightarrow F_{\alpha_2} \overset{sc}{>} F_{\alpha_1}$

A characterization of this ordering is given in the next theorem.

<u>4.2. Theorem.</u> $F \overset{sc}{>} G$ if and only if $\frac{f}{g}$ is increasing in $|t|$.

<u>Proof.</u> By Theorem 1.2, $\frac{f}{g}$ is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$.||

Thus, if we wish to show $F \overset{sc}{>} G$, we need only to consider $\frac{f}{g}$ on the positive axis. As an immediate consequence of this and Theorem 2.3, we have:

<u>4.3. Theorem.</u> Let $F_\alpha \overset{sc}{>} G_\beta$ for each pair of (α,β). Then $\int F_\alpha d\mu(\alpha) \overset{sc}{>} \int G_\beta d\nu(\beta)$ for any mixing distributions μ and ν.

Since the product of symmetric distribution functions need not be symmetric, we do not have a result analogous to Theorem 2.5. It can also be shown that this ordering is not necessarily preserved under convolution. If $F \overset{sc}{>} G$, then we can show that the even central moments of F are greater than those of G. To prove this we need the following result.

<u>4.4. Lemma.</u> (Barlow and Proschan, 1975, p. 120.)

Let $W(x)$ be a Lebesgue-Stieltjes measure, not necessarily positive for which $\int_t^\infty dW(x) \geq 0$ for all t, and let $h \geq 0$ be increasing. Then $\int_{-\infty}^{\infty} h(x)dW(x) \geq 0$.

<u>4.5. Theorem.</u> $F \overset{sc}{>} G \Rightarrow \mu_{2n}(F) \geq \mu_{2n}(G)$ for all n.

<u>Proof.</u> $F \overset{sc}{>} G \Rightarrow F(t) \leq G(t)\ \forall t \geq 0$.

Let $W(x) = \begin{cases} F(x) - G(x) & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$

and $h(x) = \begin{cases} x^{2n} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

Then by Lemma 4.4, $\mu_{2n}(F) \geq \mu_{2n}(G)$.||

4.6. Corollary. If $F \overset{sc}{>} G$, then $\text{Var } F \geq \text{Var } G$.

When F is more convex than G , G is more peaked about the origin than F. We now compare this notion of relative peakedness to the following definition given by Birnbaum (1948).

4.7. Definition. Y is more peaked than X if

$$P(|Y| \geq t) \leq P(|X| \geq t) \quad \text{for all} \quad t \geq 0 .$$

If $X(Y)$ has a symmetric distribution function $F(G)$, this is equivalent to $G(t) \geq F(t)$ for all $t \geq 0$.

4.8. Theorem. If $F \overset{sc}{>} G$, then G is more peaked than F.

Proof. Following the proof of Corollary 1.6, we have

$$\frac{F(t)-\frac{1}{2}}{G(t)-\frac{1}{2}} \uparrow \text{ in } t \quad \text{for} \quad t \geq 0 .$$

At ∞ , the limit is 1 and we have $F(t) \leq G(t)$ for all $t \geq 0$.||

We conclude this section with the following theorem.

4.9. Theorem. Suppose $F \overset{sc}{>} G$ and F is unimodal. Then G is unimodal.

Proof. Both $\frac{g}{f}$ and f are nonnegative and decreasing on $[0,\infty)$; thus g is decreasing on $[0,\infty)$.||

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Hardy, Littlewood and Pólya (1934) introduced the notion of one function being convex with respect to a second function and developed some inequalities concerning the means of the functions. We use this notion to establish a partial order called convex-ordering among functions. In particular, the distribution functions encountered in many parametric families in reliability theory are convex-ordered. We have formulated some inequalities which can be used for testing whether a sample comes from F or G, when F and G are within the same convex family. Performance characteristics of different coherent structures can also be compared with respect to this partial ordering. For example, we will show that the reliability of a k+1-out-of-n system is convex with respect to the reliability of a k-out-of-n system.

When F is convex with respect to G, the tail of the distribution F is heavier than that of G; therefore, our convex ordering implies stochastic ordering. The ordering is also related to total positivity and monotone likelihood ratio families. This provides us a tool to obtain some useful results in reliability and mathematical statistics.

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